Linear Circuits

A resistor which obeys Ohm's Law, $V = R\,I$, is plotted on a V-I diagram as a straight line which passes through the origin. Thus is shown on the figure below. Ohm's Law is a special case of the functions

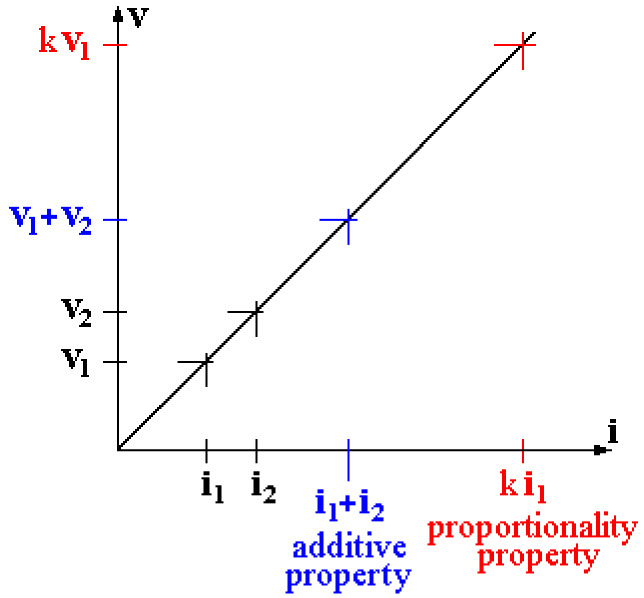
$$y_1 = f(x_1), \qquad y_2 = f(x_2)$$

such that

$$ky_1 = f(kx_1) \qquad \text{the proportionality property}$$

and

$$ky_1 + ky_2 = f(kx_1 + kx_2) \qquad \text{the additive property}$$



Circuit components which may be described by relationships of the form

$$y = ax \qquad y = b\frac{dx}{dt} \qquad y = \int cx\,dt$$

all have the properties required for linear elements. These equations are recognized as the V-I relationships for resistors, inductors and capacitors, and all circuit theorems and methods which have been derived for resistive circuits may be extended (later) to circuits which include them.

Note: Power, being proportional to voltage squared or current squared, is not represented by a linear relationship.

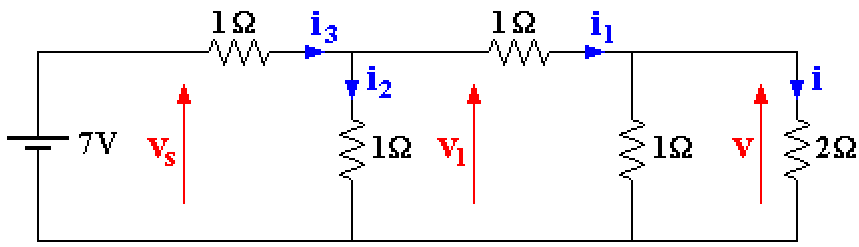
Linearity Properties and Ladder Networks

The linearity properties of circuit elements may be used to advantage in "ladder" networks. A ladder network is one where the circuit elements are progressively added in series and parallel from left to right, thus forming a chain-like series of loops.

Consider the following example which illustrates the principle for ladder networks.

Note that the rigourous solution techniques of the node voltage method and mesh analysis may also be used for ladder networks.

Example



Find the current flowing through the 2Ω resistor.

Assume that the current flowing through the resistor is **i**= 1A. The source voltage that would give this value of current will now be determined. From this value, the proportionality property is used to find the necessary current for a 7V source. Therefore the voltages and currents may be evaluate in the following order:

$$
\begin{aligned}
\mathbf{i} &= 1\text{A} \quad \text{(assumed)} \\
\mathbf{v} &= 2\text{V} \\
\mathbf{i_1} &= 1\text{A} + \frac{2\text{V}}{1\Omega} = 3\text{A} \\
\mathbf{v_1} &= 2\text{V} + 3\text{A} \times 1\Omega = 5\text{V} \\
\mathbf{i_2} &= 3\text{A} + \frac{5\text{V}}{1\Omega} = 8\text{A} \\
\mathbf{v_s} &= 5\text{V} + 8\text{A} \times 1\Omega = 13\text{V}
\end{aligned}
$$

Hence for a current **i**= 1A to flow, a source voltage of 13V is required. For the true source voltage of 7V, through the proportionality properties of a linear circuit, **i**= 7/13A, i.e.

i= 0.538 A.

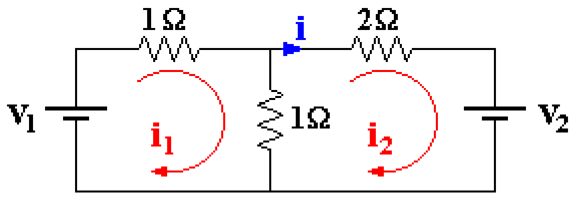
Superposition

Superposition may be used in the analysis of linear circuits which contain more than one independent source. Using the additive properties of linear circuits, the complete circuit voltage and current responses may be obtained by analyzing the circuit with several single-input responses, one at a time.
In the following example, a complete solution will be obtained first by both mesh analysis and the node voltage method. This will then be followed with a solution using the independent sources, one at a time.

Note that it is not proposed that all multi-source circuits have to be analyzed using superposition principles.

Example:

Calculate the current, **i**, in the following circuit.



Analysis – using mesh analysis

By inspection,

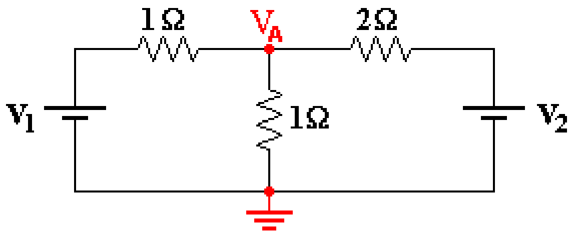
$$\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix} \quad \text{giving} \quad \begin{bmatrix} 2 & -1 \\ 0 & 2.5 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 + V_1/2 \end{bmatrix}$$

Therefore

$$\mathbf{i} = \mathbf{i_2} = 0.2V_1 - 0.4V_2 \quad \text{A}$$

Analysis – using the node voltage method

Label the two nodes as V_A and the reference node as shown in the diagram below.



Applying Kirchhoff's Current Law at the V_A node gives

$$\frac{V_A - V_1}{1} + \frac{V_A}{1} + \frac{V_A - V_2}{2} = 0, \qquad \text{i.e.} \quad V_A = 0.4V_1 + 0.2V_2$$

Now $\quad \mathbf{i} = \dfrac{V_A - V_2}{2} = 0.2V_1 + 0.1V_2 - 0.5V_2$

Therefore (in agreement with the first method)

$$\mathbf{i} = \mathbf{i_2} = 0.2V_1 - 0.4V_2 \quad \text{A}$$

Superposition

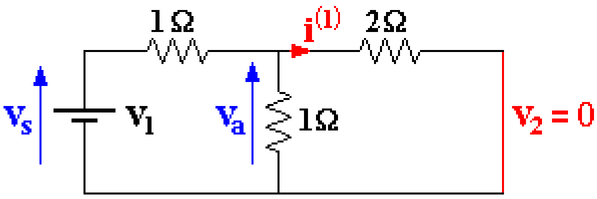
To demonstrate the principle of superposition, the component parts of the current, **i**, are found by considering the circuit with one active source at a time. The sources which are not being considered are deactivated, i.e. they remain in the circuit but are effectively turned down to zero.

A voltage source which is set to zero has no voltage across it, but may have any current flowing through it as determined by the rest of the circuit. Thus a deactivated voltage source remains in the circuit but with the properties of a short circuit across its terminals.

A current source which is set to zero has no current through it, but may have any voltage across it as determined by the rest of the circuit. Thus a deactivated current source remains in the circuit but with the properties of an open circuit across its terminals.

Important: Dependent sources cannot be deactivated or removed from the circuit at any time.

For the left-hand source:



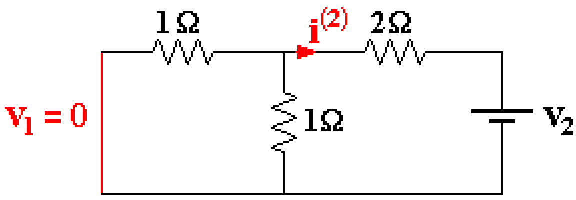
The right-hand source is deactivated, i.e. it appears as a short circuit. There are obviously many ways of determining the value of the current, e.g. by considering the circuit as a simple ladder network.

If it is assumed that $\mathbf{i}^{(1)}$= 1A, then V_a= 2V and V_1= 5V.

Therefore, using the proportionality property for a linear circuit,

$$\mathbf{i}^{(1)} = 0.2\, V_1$$

For the right-hand source:



The source see an equivalent resistance of 2.5Ω. Therefore

$\mathbf{i}^{(2)} = -0.4 V_2$ (Note the minus sign.)

Combining the results for each voltage source gives

$$\mathbf{i} = 0.2 V_1 - 0.4 V_2 \quad \mathrm{A}$$

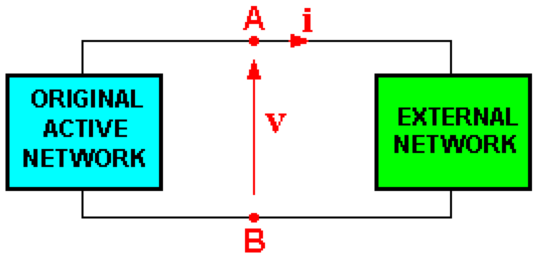
in agreement with the earlier result.

Caution:

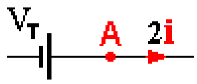
If you wish to use superposition for circuit analysis and you have to find the power dissipated in a circuit element, remember that since power is proportional to voltage squared or current squared it is not represented by a linear expression. Therefore either the voltage or current associated with the circuit element should be evaluated first, before determining the power as the final step in the calculations.

Thévenin's Theorem

Thévenin's Theorem is used in the derivation of a simplified equivalent network to replace a complete part of an active circuit. Consider a circuit which may be divided into two parts, an external network or load and the remainder of the original active network as illustrated below. The point of connection between them will be the two (and only two) lines with terminals A and B.



The principles of superposition are used to understand the correctness of the theorem as it applies to linear circuits. A new voltage of V_T is inserted into one line and is of just sufficient magnitude to double the current which flows in the circuit.



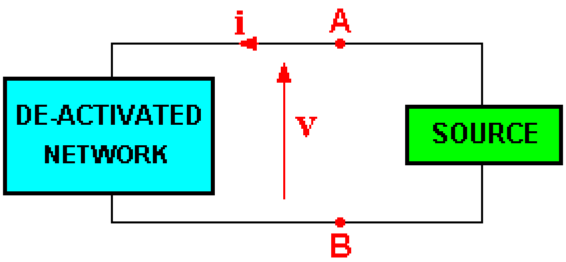
Now deactivate all the independent sources in the original active network. The current returns to its original value and the only independent source is now V_T.

For the deactivated network:

(i)
It may be replaced by its equivalent resistance, determined by series and/or parallel resistance calculations.

OR

(ii)
It may be driven by a 1A or 1V source and its terminal V-I relationship determined for the internal equivalent resistance of the deactivated network.



OR

(iii)
Without need to deactivate the network, the equivalent resistance may be determined from the open circuit voltage and short circuit current. (see a later section)

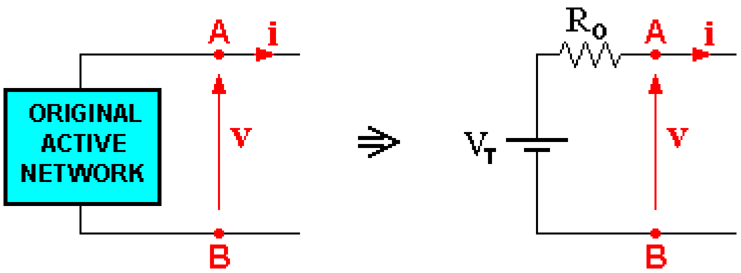
OR

(iv)
Two different value resistors may be connected as external loads and the terminal voltage measured for each case. From the voltages, the resistance may be deduced as the drop in voltage divided by the rise in current. This will be described further in the section on the terminal characteristics and is important because it forms the basis for a practical determination of the output resistance of a source.

The resistance of the deactivated network is called the Thévenin Resistance, R_{TH}. Since, later, it will be seen that this resistance is identical to the Norton Resistance, R_N, the two resistances are commonly denoted by the term R_O, implying the output resistance.

Hence Thévenin's Theorem states:

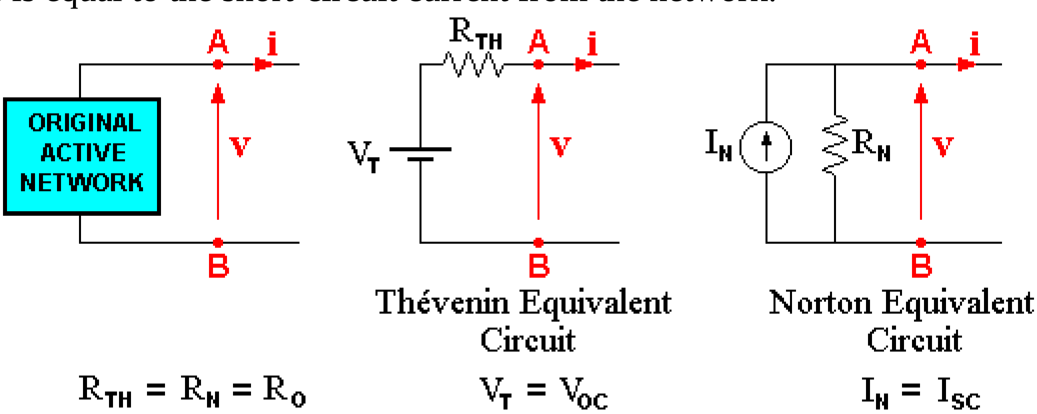
Any two-terminal linear network may be replaced by a constant voltage source $V_T = V_{OC}$ and a resistance R_O connected in series.
V_T is the voltage which would appear across the terminals if the external circuit is removed, i.e. the open circuit voltage.
R_O is the resistance measured between the two terminals if all the independent sources are deactivated and the external circuit is disconnected.



Thévenin Equivalent Norton Equivalent Circuit

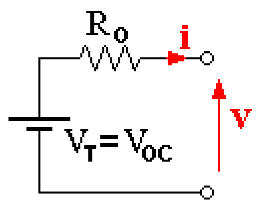
Norton's Equivalent Circuit

The Norton Equivalent Circuit is derived in a similar manner to the Thévenin Equivalent Circuit in the previous section, except that a parallel-connected current source is now inserted across the output terminals with the intention of doubling the current from the complete circuit.
Since the short-circuit currents which flow from both the original circuit and the Norton equivalent must be the same, it is clear that the Norton equivalent current source is equal to the short-circuit current from the network.



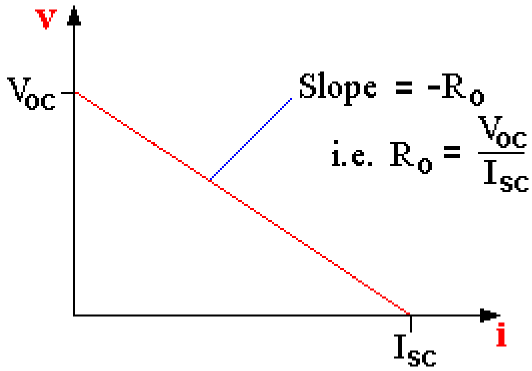
Thévenin's and Norton Circuit Terminal Characteristics

A two-terminal network may be characterized in terms of its output characteristics. Thus, as different loads are connected to the network, the voltage/current response may be determined for the output terminals.
Taking the Thévenin equivalent circuit for a network, the terminal voltage is given by



$$v = V_T - R_o i$$

Plotting this equation on a V-I diagram gives

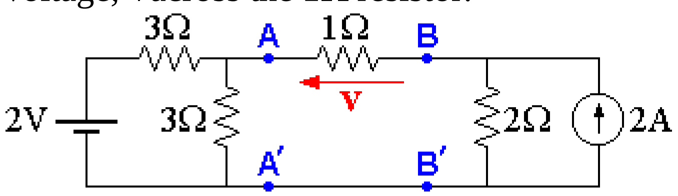


Note that this same response may also be given in terms of the Norton equivalent circuit as

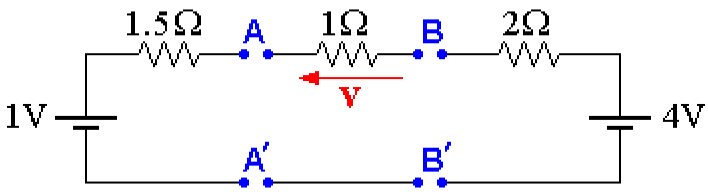
$$i = I_{sc} - \frac{v}{R_o}$$

Example:

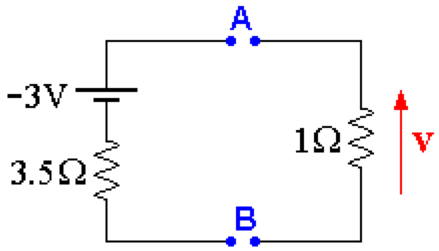
Use Thévenin's and Norton's Theorems to reduce this circuit to a simplified form and hence find the voltage, v across the 1Ω resistor.



The circuits to the left of AA' and to the right of BB' are replaced by their Thévenin equivalent circuits. Thus the circuit becomes



Grouping the elements of the two Thévenin equivalent circuits together gives



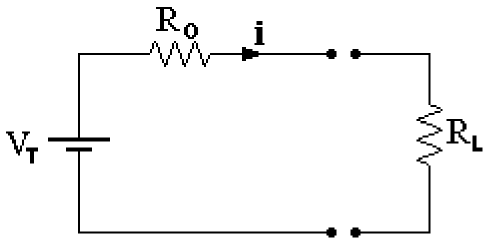
Hence, by voltage division,

$$v = -0.67 \text{ volt.}$$

Maximum Power Transfer Theorem

It is often desirable to obtain as much power as possible from a practical source. The Thévenin equivalent circuit for the source is useful for deriving the necessary circuit requirements.

The maximum power transfer theorem is considered in terms of the following circuit



with the questions:

What is the maximum power that may be delivered to the load resistance, R_L?

What is the value of the load resistance for this to occur?

The current flowing in the circuit

$$i = \frac{V_T}{R_o + R_L}$$

The power delivered to the load

$$p = i^2 R_L = V_T^2 \frac{R_L}{(R_o + R_L)^2}$$

Noting that V_T and R_o are constant, then for maximum power transfer

$$\frac{dp}{dR_L} = 0$$

i.e.

$$V_T^2 \left\{ \frac{1}{(R_o + R_L)^2} - \frac{2R_L}{(R_o + R_L)^3} \right\} = 0$$

or

$$V_T^2 \left\{ \frac{R_o - R_L}{(R_o + R_L)^3} \right\} = 0$$

For a non-trivial solution

$$R_L = R_o$$

That this is indeed a maximum may be shown since

$$\frac{d^2p}{dR_L^2} < 0$$

Intuitively, it is seen that

$p = 0$ when $R_L = 0$ and increases with R_L

$p = 0$ when $R_L = \infty$ but is finite for large R_L

i.e. p must have passed through a maximum as R_L varied through the range $0 \rightarrow \infty$.

The maximum power transfer theorem states:

The maximum power is delivered by a practical source when the load resistance is equal to the internal resistance of the source

$$\text{Show that} \quad p_{max} = \frac{V_T^2}{4\,R_L} \quad \text{when } R_L = R_o\,.$$